the temperature; s, its parameter; $I_\nu(x)$, the Bessel function with imaginary argument of order ν; $K_\nu(x)$, the MacDonald function of order ν; ϑ and Θ, dimensionless temperature; Po, Pomerantz number; Bi, Biot number; Fo, Fourier's number; ρ, dimensionless polar radius; b_1^*, dimensionless radius of the circle on which the inclusions are placed; R^*, dimensionless radius of the plate.

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NUMERICAL METHOD FOR SOLVING HEAT-CONDUCTION PROBLEMS FOR TWO-DIMENSIONAL BODIES OF COMPLEX SHAPE

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A finite-difference scheme is described for a curvilinear orthogonal net which permits the use of a single algorithm for calculating bodies of various shapes.

The construction of curvilinear difference nets by calculating the conformal mapping of a canonical region (rectangle) into the given region was described in [1]. Unlike a rectangular net which is typical for the finite-difference method [2-4] and in practically important cases bears little resemblance to the boundaries of the body, an orthogonal net reflects the nature of the boundary, and has no nonregular nodes. An orthogonal net is more convenient to work with than the nets commonly used in the method of finite elements [5] since all quantities referring to the nodes of such a net (e.g., their coordinates) can be written in the form of a rectangular matrix. Using equations of the elliptic type as an example, variational-difference schemes for such nets were discussed in [6].

An analysis in [7] showed that finite-difference schemes have distinct advantages over variational-difference schemes in solving heat-conduction problems. For this reason the finite-difference method is of great interest for solving heat-conduction problems with orthogonal nets [8]. The practical use of the algorithm obtained confirmed its adequate accuracy, high speed, and, what is particularly important, the simplicity of its application for bodies of various shapes. However, it is not clear from [8] under what conditions and at what rate the scheme converges to the solution of the original equation.

We describe a procedure which employs a set of standard programs to automate the process of solving the heat-conduction equation for a broad class of two-dimensional regions. If an orthogonal net is constructed by conformal mapping, the rate of convergence of the finite-difference scheme can be estimated.

Let the function $F(w) = F(u + iv)$ map the rectangle G conformally into the region G^* in the (x, y) plane. We assume that $F(u + iv)$ is known and that $\partial F/\partial w$ exists and is finite on the boundary Γ of the rectangle. The latter implies that G^* is a curvilinear quadrangle in which all the angles are right angles.

We consider the problem for the heat-conduction equation posed in G^*:

$$c\frac{\partial T}{\partial \tau} - LT = Q, \tag{1}$$

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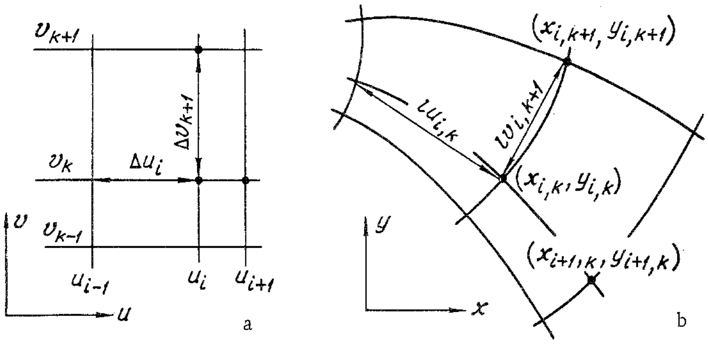

Fig. 1. Rectangular (a) and curvilinear (b) nets.

$$T|_{\Gamma^*} = f(x, y, \tau), \quad T(x, y, 0) = T_0(x, y).$$

Here Γ^* is the boundary of region G^*; $LT = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right)$; $\lambda = \lambda(x, y, \tau)$; $c = c(x, y, \tau) > 0$; $Q = Q(x, y, \tau)$.

In the coordinates u, v Eq. (1) can be written in the form

$$c \frac{\partial \Theta}{\partial \tau} - \frac{1}{g}(L_1 + L_2)\Theta = Q, \tag{2}$$

$$\Theta|_{\Gamma} = f(u, v, \tau), \quad \Theta(u, v, 0) = T_0(u, v).$$

Here Γ is the boundary of the rectangle G; $L_1\Theta = \frac{\partial}{\partial u}\left(\lambda \frac{\partial \Theta}{\partial u}\right)$; $L_2\Theta = \frac{\partial}{\partial v}\left(\lambda \frac{\partial \Theta}{\partial v}\right)$; $g(u, v) = \left|\frac{\partial F(w)}{\partial w}\right|^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial x}{\partial v}\right)^2 > 0.$

Assuming that problems (1) and (2) have a unique solution,

$$\Theta(u, v) = T(x(u, v), y(u, v)). \tag{3}$$

We construct in G the net $\{(u = u_i, v = v_k), i = 0, 1 \ldots N, k = 0, 1 \ldots M\}$ shown in Fig. 1. In mapping the rectangle G into G* the point (u_i, v_k) is mapped into the point $(x(u_i, v_k)\ y(u_i, v_k))$ and the side of the rectangle $\Delta u_i = \{(u_{i-1}, v_k), (u_i, v_k)\}$ is mapped into an arc which subtends the chord $lu(u_i, v_k, \Delta u_i) = \sqrt{g(\bar{u}_i v_k)}\,\Delta u_i + O(\Delta u_i^3)$.

The side of the rectangle $\Delta v_k = \{(u_i, v_{k-1}), (u_i, v_k)\}$ is mapped into an arc with the chord

$$lv(u_i, v_k, \Delta v_k) = \sqrt{g(u_i, \bar{v}_k)}\,\Delta v_k + O(\Delta v_k^3).$$

Here $\bar{u}_i = (u_{i-1} + u_i)/2$; $\bar{v}_k = (v_{k-1} + v_k)/2$.

We write for (2) a locally one-dimensional scheme [9], and solve in succession the equations

$$\begin{aligned} c^{j+1}\frac{\theta_1^{j+1} - \theta_1^j}{\Delta\tau} - D\Lambda_1\theta_1^{j+1} &= F^{j+1/2}, \\ c^{j+1}\frac{\theta_2^{j+1} - \theta_2^j}{\Delta\tau} - D\Lambda_2\theta_2^{j+1} &= F^{j+1/2} \end{aligned} \tag{4}$$

with the initial conditions $\theta_1^0 = T_0(u, v)$, $\theta_1^j = \theta_2^j$, $j = 1, 2, \ldots$, $\theta_2^j = \theta_1^{j+1}$, $j = 0, 1, 2 \ldots$. The boundary conditions for θ_1^j are the values of $f(u, v, \tau^j)$ on the sides of the rectangle G perpendicular to the u axis, and for θ_1^j the values of $f(u, v, \tau^j)$ on the sides perpendicular to the v axis. The solution of the problem is by definition the element $\theta^j = \theta_2^j$. Here $\theta^j \sim \Theta$; $C^j \sim c$; $F^j \sim Q$; $D \sim 1/g$ are the values of the network functions at the time $\tau_j = \Delta\tau_j$. We denote the coefficients of the difference operators $\Lambda_1 \sim L_1$ and $\Lambda_2 \sim L_2$ as follows:

$$A_{i,k}^j = \lambda(\bar{u}_i, v_k, \tau_j), \quad B_{i,k}^j = \lambda(u_i, \bar{v}_k, \tau_j),$$

$$\overline{\Delta u}_i = (\Delta u_i + \Delta u_{i+1})/2, \quad \overline{\Delta v}_k = (\Delta v_k + \Delta v_{k+1})/2.$$

If $\theta(u, v, \tau)$ and $Q(u, v, \tau)$ are sufficiently smooth [9], the locally one-dimensional scheme (4) converges uniformly to the solution of problem (2) at a rate $O(h^2 + \Delta\tau)$, $h = \max(\Delta u_i, \Delta v_k)$, $1 \le i \le N$, $1 \le k \le M$.

We now assume that we know only the coordinates of the nodes of the orthogonal net $\{x_{i,k}, y_{i,k}\}$. Let

$$lu_{i,k} = \sqrt{(x_{i,k} - x_{i-1,k})^2 + (y_{i,k} - y_{i-1,k})^2},$$
$$lv_{i,k} = \sqrt{(x_{i,k} - x_{i,k-1})^2 + (y_{i,k} - y_{i,k-1})^2},$$
$$\overline{lu}_{i,k} = (lu_{i-1,k} + lu_{i,k})/2, \quad \overline{lv}_{i,k} = (lv_{i,k-1} + lv_{i,k})/2,$$
$$l = \max(lu_{i,k}, lv_{i,k}), \quad 1 \leqslant i \leqslant N, \quad 1 \leqslant k \leqslant M.$$

We require that the conditions

$$lu_{i,k} = lu(u_i, v_k, \Delta u_i) + O(\Delta u_i^2 + \Delta v_k^2),$$
$$lv_{i,k} = lv(u_i, v_k, \Delta v_k) + O(\Delta v_k^2 + \Delta u_k^2)$$

be satisfied. For this it is sufficient that

$$x_{i,k} = x(u_i, v_k) + O(\Delta u_i^2 + \Delta v_k^2),$$
$$y_{i,k} = y(u_i, v_k) + O(\Delta u_i^2 + \Delta v_k^2). \tag{5}$$

Let us consider scheme (4) with perturbed coefficients:

$$A^{*j}_{i,k} = \lambda\left(\frac{x_{i,k} + x_{i-1,k}}{2}, \frac{y_{i,k} + y_{i-1,k}}{2}, \tau_j\right) \frac{\Delta u_i}{\overline{\Delta v}_k} \frac{\overline{lv}_{i,k} + \overline{lv}_{i-1,k}}{2lu_{i,k}} = A^j_{i,k}(1 + O(l)),$$

$$B^{*j}_{i,k} = \lambda\left(\frac{x_{i,k} + x_{i,k-1}}{2}, \frac{y_{i,k} + y_{i,k-1}}{2}, \tau_j\right) \frac{\Delta v_k}{\overline{\Delta u}_i} \frac{lu_{i,k} + lu_{i,k-1}}{2lv_{i,k}} = B^j_{i,k}(1 + O(l)),$$

$$C^{*j}_{i,k} = c(x_{i,k}, y_{i,k}, \tau_j) = C^j_{i,k}(1 + O(l^2)),$$
$$F^{*j}_{i,k} = Q(x_{i,k}, y_{i,k}, \tau_j) = F^j_{i,k}(1 + O(l^2)),$$
$$D^*_{i,k} = \frac{\overline{\Delta u}_i\, \overline{\Delta v}_k}{\overline{lu}_{i,k}\, \overline{lv}_{i,k}} = D_{i,k}(1 + O(l)).$$

After some simple transformations we obtain instead of (4)

$$C^{*j+1}\frac{t_1^{*j+1} - t_1^{*j}}{\Delta\tau} - \tilde{D}_1\tilde{\Lambda}_1 t_1^{*j+1} = F^{*j+1}/2,$$
$$C^{*j+1}\frac{t_2^{*j+1} - t_2^{*j}}{\Delta\tau} - \tilde{D}_2\tilde{\Lambda}_2 t_2^{*j+1} = F^{*j+1}/2. \tag{6}$$

We specify the initial data in the form $t_2{}^{*j} = t_1{}^{*j+1}$, $j = 0, 1, 2\ldots$, $t_1{}^{*0} = T_0(x_{i,k}, y_{i,k}) = \theta_1^0(1 + O(l^2))$, $t_1{}^{*j} = t_2{}^{*j}$, $j = 1, 2\ldots$ The boundary values for $t_1{}^{*j}$ and $t_2{}^{*j}$ are specified on pairs of opposite sides of the curvilinear quadrangle by the function $f(x, y, \tau)$. Because of (5) $f(x_{i,k}, y_{i,k}, \tau) = f(u_i, v_k, \tau)(1 + O(l^2))$. Here

$$\tilde{D}_1 = 1/\overline{lv}_{i,k}; \quad \tilde{D}_2 = 1/\overline{lu}_{i,k};$$

$$\tilde{\Lambda}_1 t^{*j}_{i,k} = \frac{1}{\overline{lu}_{i,k}}\left(\tilde{A}^j_{i+1,k}\frac{t^{*j}_{i+1,k} - t^{*j}_{i,k}}{lu_{i+1,k}} - \tilde{A}^j_{i,k}\frac{t^{*j}_{i,k} - t^{*j}_{i-1,k}}{lu_{i,k}}\right);$$

$$\tilde{\Lambda}_2 t^{*j}_{i,k} = \frac{1}{\overline{lv}_{i,k}}\left(\tilde{B}^j_{i,k+1}\frac{t^{*j}_{i,k+1} - t^{*j}_{i,k}}{lv_{i,k+1}} - \tilde{B}^j_{i,k}\frac{t^{*j}_{i,k} - t^{*j}_{i,k-1}}{lv_{i,k}}\right);$$

$$\tilde{A}^j_{i,k} = \lambda\left(\frac{x_{i,k} + x_{i-1,k}}{2}, \frac{y_{i,k} + y_{i-1,k}}{2}, \tau_j\right)(\overline{lv}_{i,k} + \overline{lv}_{i-1,k})/2;$$

$$\tilde{B}^j_{i,k} = \lambda\left(\frac{x_{i,k} + x_{i,k-1}}{2}, \frac{y_{i,k} + y_{i,k-1}}{2}, \tau_j\right)(\overline{lu}_{i,k} + \overline{lu}_{i,k-1})/2.$$

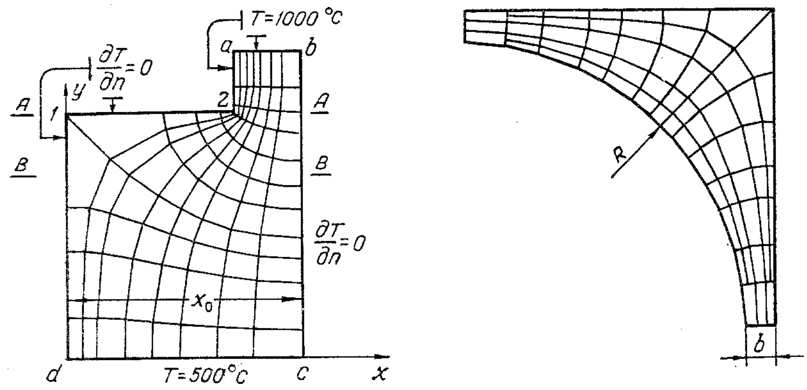

Fig. 2. Examples of the construction of an orthogonal net.

Thus, the coefficients in Eq. (6) are uniquely determined if the coordinates of the nodes of the orthogonal net $\{x_{i,k}, y_{i,k}\}$ are known. This permits considering (6) formally as a locally one-dimensional scheme approximating the original Eq. (1) by an orthogonal net constructed in the curvilinear quadrangle G*. Here $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ are positive self-adjoint operators which ensure the unconditional stability of the implicit scheme (6).

In the class of functions Θ having continuous derivatives $\partial^2\Theta/\partial u^2$ and $\partial^2\Theta/\partial v^2$, the following estimates hold:

$$\tilde{D}_1\tilde{\Lambda}_1\Theta = D\Lambda_1\Theta + O(l), \qquad \tilde{D}_2\tilde{\Lambda}_2\Theta = D\Lambda_2\Theta + O(l),$$

which enable one to prove the uniform convergence of t^* to θ at a rate of $O(l)$, and consequently also the convergence of t^* to the solution of problem (2) at the rate $O(l + \Delta\tau)$. Taking account of (3) and (5), the uniform convergence of t^* and Θ implies the uniform convergence of t^* to the solution for T of the original problem (1) at the rate $O(l + \Delta\tau)$:

$$\|T^j - t^{*j}\|_C \leqslant M(l + \Delta\tau), \quad j = 1,\ 2\ldots, \tag{7}$$

where $l = \max(l u_{i,k};\ l v_{i,k})$, $1 \le i \le N$, $1 \le k \le M$.

The procedure described can be used in other schemes (e.g., longitudinal-transverse) and for other boundary conditions, and enables one to obtain their analogs on an orthogonal net. In the case of a locally one-dimensional scheme it is possible to treat more complicated regions G consisting of curvilinear quadrangles G_i^* in each of which its own orthogonal net is constructed. The transformation to an orthogonal net is always considered as a small perturbation of the coefficients of the original scheme.

It can be required of any scheme that the solution depend continuously on the perturbation of the coefficients of the difference operator (the coefficient stability of the scheme). Hence it follows that the analog of the scheme converges uniformly on an orthogonal net.

At the present time methods are being developed for constructing curvilinear nets in a preassigned region [10]; numerical methods of conformal mappings are also known [11]. Therefore, such a procedure enables one to automate the solution of the heat-conduction equation in a complex region.

We note that the construction of an orthogonal net in a region is not unique; it can be constructed by taking account of the special features of the problem. An interesting example of this is the net in the quadrangle abcd shown in Fig. 2. In the neighborhood of point 2 the nodes are close together, which enables one to obtain detailed information about the temperature distribution at this location. At point 1 the nodes of the net are far apart, but it is known beforehand that the temperature gradients here are not large.

Calculations for such a region by the conventional locally one-dimensional scheme for a rectangular net (933 nodes) and by our method for an orthogonal net (964 nodes) were compared. Figure 3a compares the solutions of the nonlinear heat-conduction equation with zero initial conditions. The boundary conditions are shown in Fig. 2a. Because of the strong temperature dependence of λ and ρc (Fig. 3b), the solution of the problem resembles thermal waves. It is clear that both methods give practically the same result, but interpolation of the temperature in the neighborhood of point 2 from its known values at the nodes is more reliably performed with the orthogonal net, since it has a high density of nodes in this region. In this case the computational times

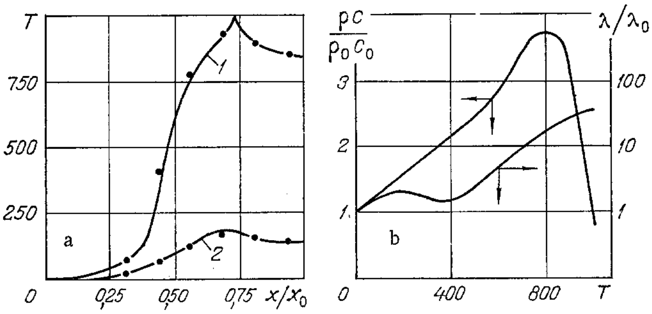

Fig. 3. Comparison of calculations for a step region: a) temperature distribution in cross sections AA (curve 1) and BB (curve 2) for $Fo = \lambda_0 \tau / \rho_0 c_0 x_0^2 = 0.01$; curves calculated by our method (964 nodes of an orthogonal net), points calculated by conventional method (933 nodes of a rectangular net); b) dependence of thermal conductivity $\lambda(T)/\lambda(T = 0)$ and volumetric heat capacity $\rho c(T)/\rho c(T = 0)$ on temperature T in °C.

for an orthogonal and a rectangular net turn out to be practically the same. Thus, even when the region is subdivided into rectangular elements, the use of an orthogonal net may be expedient if, for example, it is necessary to bunch the nodes of the net near a singular point.

The main advantage of the method described is that in switching from one body to another the same program can be used, whereas with a rectangular net the whole program must be practically rewritten each time.

There is a broad class of bodies for which an orthogonal net can be constructed without difficulty. Figure 2b shows an orthogonal net for an angle bracket. A net of this type can also be employed for an I-beam which can be formed from such brackets. This net was constructed by using a set of standard programs for the numerical construction of a conformal mapping by trigonometric interpolation. These same standard programs can be used to construct orthogonal nets for almost all sections of the metallurgical industry, which in the last analysis permits the complete automation of the calculation of heating and cooling of various sections.

NOTATION

x, y, independent variables; u, v, orthogonal coordinates; $F(w) = F(u + iv)$, function of a complex variable; $g(u, v) = \| \partial F(w)/\partial w \|$, Jacobian of transformation from (u, v) to (x, y); λ, thermal conductivity; c, volumetric heat capacity; Q, heat release per unit volume; T, temperature; f, value of temperature on boundary of region; τ, time; L, L_1, L_2, differential operators; $\Theta(u, v)$, solution of differential problem in canonical region; θ^j, θ_1^j, θ_2^j, t^j, t_1^j, t_2^j, network functions in canonical region; θ^j, t^{*j}, solutions of difference problems using rectangular and orthogonal nets respectively; $\{u_i, v_k\}$, rectangular net in canonical region G; $\{x_{i,k}, y_{i,k}\}$, orthogonal net in given region G*; Δu_i, Δv_k, dimensions of cell of rectangular net; $l u_{i,k}$, $l v_{i,k}$, dimensions of cell of orthogonal net; h, l, maximum dimension of cell for rectangular and orthogonal nets respectively; Λ_1, Λ_2, $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, difference operators for rectangular and orthogonal nets; A, B, C, D, A*, B*, C*, D*, coefficients of difference scheme for rectangular net; $\tilde{D}$, $\tilde{A}$, $\tilde{B}$, coefficients of difference scheme for orthogonal net.

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ANALYTIC SOLUTIONS OF PARABOLIC AND HYPERBOLIC HEAT-TRANSFER EQUATIONS FOR NONLINEAR MEDIA

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New classes of analytic solutions are obtained which describe unsteady temperature distributions and take account of the temperature dependence of the thermophysical properties of the material. The concept of a solution of the boundary layer transition type is introduced for the generalized heat-transfer equation.

We consider the nonlinear heat-conduction equation in a one-dimensional plane region

$$c(T)\,T_t = [\lambda(T)\,T_x]_x. \tag{1}$$

We introduce a new function $\xi = \xi(x, t)$ with the following properties:

$$\xi_x = U(T), \quad \xi_t = \lambda T_x,$$
$$U(T) = U_0 + \int_0^T c(T)\,dT, \quad U_0 \equiv \text{const}.$$

We change from the variables (x, t) to new independent variables (ξ, t):

$$d\xi = U(T)\,dx + (\lambda U T_\xi)\,dt,$$
$$D(\xi,\ t)/D(x,\ t) = U \neq 0,$$

so that the initial Eq. (1) takes the form

$$\beta(T)\,T_t = [\lambda(T)\,T_\xi]_\xi,\ \beta = cU^{-2},\ T = T(\xi,\ t), \tag{2}$$

where the Cartesian coordinate is related to the new variable by the equation

$$x(\xi,\ t) = \int \frac{d\xi}{U(\xi,\ 0)} - \int_0^t \lambda T_\xi dt, \quad U = U[T(\xi,\ t)]. \tag{3}$$

A comparison of Eqs. (1) and (2) shows that to each one-dimensional unsteady temperature distribution in a medium with the thermophysical parameters c(T) and λ(T) there corresponds a certain one-dimensional unsteady temperature distribution in a medium with volumetric heat capacity β(T) and a thermal conductivity

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